MONOTONE PERTURBATIONS OF THE LAPLACIAN IN $L^1(\mathbf{R}^N)$

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ABSTRACT

The semilinear perturbation of Poisson's equation (E): $-\Delta u + \beta(u) \ni f$, where β is a maximal monotone graph in $\mathbf{R}$, has been investigated by Ph. Bénilan, H. Brézis and M. Crandall for $f \in L^1(\mathbf{R}^N)$, $N \geqq 1$, under the assumptions $0 \in \beta(0)$ if $N \geqq 3$ and $0 \in \beta(0) \cap \operatorname{Int} \beta(\mathbf{R})$ if $N = 1,2$. We discuss in this paper the solvability and well-posedness of (E) in terms of any maximal monotone graph β. In particular, if β takes only positive values and $N \geqq 3$ we prove that no solution exists; if $N = 2$ we give necessary and sufficient conditions on β and f for (E) to be solvable in a natural sense.

Introduction

This paper deals with the following nonlinear perturbation of Poisson's equation in $\mathbf{R}^N$, $N \geqq 1$:

$$(\mathrm{E}) = (\mathrm{E}_{\beta,f,N}) \qquad -\Delta u + \beta(u) \ni f$$

where β is a maximal monotone graph in $\mathbf{R}$ (m.m.g. for short; see e.g. Brézis [4]). In particular you can assume throughout that β is a continuous, nondecreasing function on $\mathbf{R}$ and write (E) as an equality. The function f will belong to $L^1(\mathbf{R}^N)$.

In case $0 \in \beta(0)$ if $N \geqq 3$ or $0 \in \beta(0) \cap \operatorname{Int} \beta(\mathbf{R})$ if $N = 1,2$ we owe Bénilan, Brézis and Crandall [2] a detailed study of (E): for every $f \in L^1(\mathbf{R}^N)$ they obtain a solution $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$ such that $w \equiv \Delta u + f \in \beta(u)$ a.e.[†] is integrable. If u is chosen in the appropriate class (see discussion to follow) w is uniquely determined and the map $T: f \to w$ is an ordered contraction in $L^1(\mathbf{R}^N)$; we say that the operator $A = -\Delta \circ \beta^{-1}$ (defined by $Aw = -\Delta u$) is *m-accretive* in $L^1(\mathbf{R}^N)$. This is an important property for then A generates a semigroup of contractions in $L^1(\mathbf{R}^N)$ that enables us to solve the evolution equation

[†] $w = \beta(u)$ if β is a function.

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$$(\mathrm{P}_{ev})\qquad \begin{cases} u_t - \Delta\beta^{-1}(u) = 0 & \text{in } \mathbf{R}^N \times]0, T[\\ u(0,x) = u_0(x) \in L^1(\mathbf{R}^N) \end{cases}$$

via discretization in time (Crandall–Liggett's Theorem, see [5], [7]).

In [6] Crandall and Evans consider equation (E) in dimension $N = 1$ for β a m.m.g. such that $\beta(\mathbf{R}) \subset (0,\infty)$, imposing conditions at infinity $u'(\pm\infty) = 0$. Attention to this case had been drawn by T. Kurtz [10] in the particular form $e^u - u'' = f$, $u'(\pm\infty) = 0$. Crandall and Evans show that the problem is solvable only if β is integrable at $-\infty$ and in that case A is accretive and $R(I + A) = L^1_+(\boldsymbol{R}) = \{f \in L^1(\mathbf{R}) : \int f > 0\} = D(A)$ so that (P_{ev}) may be solved as hinted above for initial data $u_0 \in L^1_+(\mathbf{R})$.

Bénilan and Brézis [1] consider (E) for non-negative β, $\beta(\mathbf{R}) \ni 0$, in $N \geqq 3$ when studying the Thomas–Fermi equation.

S. Fisher considers in [8] a more general situation where β is a continuous function, not necessarily monotone ($N = 1$).

Our concern in this paper is to complete the discussion of equation (E) under suitable conditions at infinity (if any needed) in terms of all possible m.m.g. β and for $f \in L^1(\mathbf{R}^N)$, $N \geqq 1$:

$N = 2$. The main novelty appears when β takes on only positive (or only negative) values. We then pose the problem consisting of equation (E) plus the condition of *null flux*:

$$\int \Delta u \cdot dx = \lim_{r\to\infty} \int_0^{2\pi} \frac{\partial u}{\partial r} r d\theta = 0$$

and prove the following result (see Theorem 2.1):

Main Result. *Let $\beta(\mathbf{R}) \subset (0,\infty)$. For every $f \in L^1(\mathbf{R}^2)$ such that $\int f > 0$ there exists a unique $u \in W^{1,1}_{loc}(\mathbf{R}^2)$ with $|\mathrm{grad}\, u| \in M^2(\mathbf{R}^2)$,*† *$\Delta u \in L^1(\mathbf{R}^2)$ and $\int \Delta u \cdot dx = 0$ such that $\Delta u + f \in \beta(u)$ a.e. if and only if β satisfies the following condition of integrability at $-\infty$:*

$$(\mathrm{B}_2)\qquad \begin{gathered} \textit{There exists } a \in \mathbf{R} \textit{ with } (-\infty, a) \subset D(\beta) \textit{ and} \\ \int_{-\infty}^{a} \beta(t)\exp(-bt)dt < \infty \quad \textit{for every } b > 0. \end{gathered}$$

$N = 3$. We prove a *negative* result in case $\beta(\mathbf{R}) \not\ni 0$: Under mild (and natural)

† $M^p(\mathbf{R}^N) = L(p,\infty)$, $1 < p < \infty$, denotes the Marcinkiewicz space or weak-L^p space, see appendix of [2].

conditions on u *no solution* of (E) can exist such that $\beta^0(u)^\dagger \in L^1(\mathbf{R}^N)$ (Proposition 4.1).

$N = 1$. [2] and [6] already contain the basic results.

It turns out that the situation is very similar when $N = 1$ and 2 and, on the other hand, for all $N \geqq 3$. For that reason our work owes much in motivation and techniques to [6] and, of course, to [2].

The plan is as follows: we collect some preliminary results and notation and formulate the problem in Section 1. In Section 2 we consider the case $N = 2$, $\beta(\mathbf{R}) \subset (0,\infty)$. In Section 3 we present a *general discussion* for $N = 1$ and 2. $N = 3$ is treated in Section 4. We conclude with a technical appendix.

1. Preliminaries

We begin with some basic definitions and notation. Given β a m.m.g. in $\mathbf{R}$ and $f \in L^1(\mathbf{R}^N)$ we say that $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ is a *solution* of the equation (E): $-\Delta u + \beta(u) \ni f$, if $\Delta u \in L^1(\mathbf{R}^N)$, in the sense of distributions and $w \equiv \Delta u + f \in \beta(u)$ a.e. In order to obtain a *well-posed problem* $(\mathrm{P}) = (\mathrm{P}_{\beta f})$, a condition on the behaviour of the solutions at infinity must be added in general. Then a subspace $\mathfrak{L}$ of $L^1_{\mathrm{loc}}(\mathbf{R}^N)$ is considered where the solutions of (P) are to be found. ($\mathfrak{L}$ contains, in general, the conditions at infinity.) We define the *solution maps* $G = G_\beta : L^1(\mathbf{R}^N) \to \mathfrak{L}$, $G(f) = u$ solution of $(\mathrm{P}_{\beta,f})^{\dagger\dagger}$ and $T = T_\beta : L^1(\mathbf{R}^N) \to L^1(\mathbf{R}^N)$, $T(f) = w \equiv \Delta u + f$ with domain $D(G) = D(T) = \{f \in L^1(\mathbf{R}^N) : (\mathrm{P}_{\beta,f})$ admits a solution in $\mathfrak{L}\}$.

We say that (P) is *well-posed from* $V \subset L^1(\mathbf{R}^N)$ *into* $\mathfrak{L}$ if

(I) $D(G) = V$ (Existence)

(II) T is single-valued (Uniqueness)

(III) $\int (Tf - T\hat{f})^+ \leqq \int (f - \hat{f})^+$ for $f, \hat{f} \in L^1(\mathbf{R}^N)$ (Continuous Dependence)

If $V = L^1(\mathbf{R}^N)$ we simply say that (P) is *well-posed in* $\mathfrak{L}$, in accordance with [2]. A *solution* of (P) will be a solution of (E) in the class $\mathfrak{L}$.

We follow, in general, notations in [2]. In particular we set

$$\mathfrak{J}_0 = \{j : \mathbf{R} \to [0,\infty) : j \text{ convex, lower semi-continuous}, j(0) = 0\},$$

$$\mathfrak{P} = \{p \in C^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) : p \text{ nondecreasing}\},$$

$$\mathfrak{P}_0 = \{p \in \mathfrak{P} : p(0) = 0\}, \qquad \mathfrak{P}_+ = \{p \in \mathfrak{P} : p \geqq 0\}.$$

We write $[u > a]$ for $\{x \in \mathbf{R}^N : u(x) > a\}$ and so on. If $\Omega \subset \mathbf{R}^N$ is measurable

† If β is multivalued and $s \in D(\beta)$, $\beta^0(s)$ is the element in $\beta(s)$ of minimal absolute value (see [4]). For a function it is simply $\beta(s)$.

†† Note that G is not necessarily single-valued.

$|\Omega| = ms\,\Omega$ is its (Lebesgue) measure. b_N is the volume of the unit ball $B_1(0)$ in $\mathbf{R}^N$, $\omega_N = Nb_N$ the $N-1$ dim. measure of the sphere $S_1(0)$. We shall use the fundamental solutions of $-\Delta$ in $\mathbf{R}^N$ (see [2], appendix): if $N \geqq 3$,

$$E_N(x) = [(N-2)b_N]^{-1}|x|^{2-N}; \quad E_2(x) = -(2\pi)^{-1}\lg|x|, \quad E_1(x) = -\tfrac{1}{2}|x|.$$

We consider several types of convergence at infinity, i.e. as $|x| \to \infty$, for functions $u = u(x)$ defined on $\mathbf{R}^N$:

(i) we say that u converges to $a \in \mathbf{R}$ *in measure*, $u(x) \to a$ (m), if for every $\varepsilon > 0$, $ms[|a-u| > \varepsilon]$ is finite. $u(x) \to \infty$ (m) if for every $a \in \mathbf{R}$, $ms[u < a] < \infty$. Similarly $u \to -\infty$ (m).

(ii) We say that $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ tends to a at infinity in *absolute integral mean*, $u \to a$ (a.i.m.) if

$$\lim_{R\to\infty} \frac{1}{|B_R|}\int_{B_R} |u(x) - a|\,dx = 0 \tag{1.1}$$

or equivalently if

$$\lim_{n\to\infty} \int_{1\leqq|x|\leqq 2} |u(nx) - a|\,dx = 0. \tag{1.2}$$

(This formulation is used in [2], appendix.)

(iii) $\lim_{x\to\infty} u = a$ in *integral mean* (i.m.) if

$$\lim_{R\to\infty} \frac{1}{|B_R|}\int_{B_R} u(x)\,dx = a. \tag{1.3}$$

Corresponding definitions hold for $|u| \to \infty$ or $u \to \pm\infty$ (i.m.). Functions in $L^p(\mathbf{R}^N)$, $1 \leqq p < \infty$ or in $M^p(\mathbf{R}^N)$, $1 < p < \infty$ tend to zero at infinity (m) and (a.i.m.).

(iv) For $N \geqq 2$ and $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ we define the angular mean of u, $\tilde{u} :]0,\infty[\to \mathbf{R}$, $\tilde{u}(r) = |\omega_N|^{-1}\int_{\sigma\in S_1} u(r\sigma)\,d\sigma$ where $d\sigma$ is the surface measure on S_1. u converges to $a \in \mathbf{R}$ at infinity *in angular mean* if $\lim_{r\to\infty}\tilde{u}(r) = a$.

2. $N = 2$, $\beta(\mathbf{R}) \subset (0,\infty)$

In this section (E): $-\Delta u + \beta(u) \ni f$ is studied for $N = 2$ and β a positive-valued m.m.g. We begin by discussing the functional setting in which (E) becomes a well-posed problem.

First, if $u \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$ is a solution of (E), then $\lim_{s\to-\infty}\beta(s) = 0$ and $\lim_{x\to\infty} u(x) = -\infty$ in measure. This is a consequence of $w \equiv \Delta u + f \in \beta(u)$ a.e. and $w \in L^1(\mathbf{R}^N)$. Hence we assume hereafter that $\beta(-\infty) = 0$.

Also if $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$ with $\Delta u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ it is easy to prove that $\tilde{u} \in C^1(]0,\infty[)$ and

$$(2.1) \quad 2\pi r\tilde{u}'(r) = r\frac{d}{dr}\left(\frac{1}{r}\int_{|x|=r} u(x)d\sigma\right) = \int_{|x|\leqq r} \Delta u \cdot dx, \qquad \text{for a.e. } r>0$$

(see e.g. [12]). As $\Delta u = f - w$, we have $\lim_r r\tilde{u}'(r) = \int_{\mathbf{R}^2}(f-w)$, as $r\to\infty$, so that $\int f = \int w$ if and only if $\lim_{r\to\infty} r\tilde{u}'(r) = 0$. This was precisely the situation in case $0\in\beta(0)\cap \mathrm{Int}\,\beta(\mathbf{R})$ (theorem 3.1 of [2]).

In our case a condition at infinity must be specified. We select expression (2.1) and interpret it as the flux emerging through $|x| = R$ by virtue of the following argument: if $u\in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2)$ with $\Delta u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ then on a.e. sphere $S_R = [|x| = R]$, $\partial u/\partial r \in L^1(S_R)$ and

$$\int_{|x|\leqq R} \Delta u \cdot dx = \int_{|x|=R} \frac{\partial u}{\partial r}\cdot d\sigma = \text{flux of grad } u \text{ through } S_R.$$

We shall solve equation (E) plus the condition of "null flux at infinity", a homogeneous-Neumann-type condition. Thus we formulate

$$(\mathrm{P}) \qquad \begin{cases} -\Delta u + \beta(u) \ni f, \\ \int \Delta u \cdot dx = 0. \end{cases}$$

We remark that a condition of positive flux at infinity would not be compatible with our problem in general. In fact we have

PROPOSITION 2.1. *Let $u\in L^1_{\mathrm{loc}}(\mathbf{R})$ be any function with $\Delta u \in L^1(\mathbf{R})$. If a section w of $\beta(u)$ is integrable and either* (i) *$D(\beta)$ is bounded above or $D(\beta) = \mathbf{R}$ and* $\liminf_{r\to\infty}\beta(r)/r > 0$ *or* (ii) $|\mathrm{grad}\, u| \in M^2(\mathbf{R}^2)$, *then necessarily* $\int \Delta u \cdot dx \leqq 0$.

PROOF. Observe first that $\int \Delta u > 0$ means $\lim_{r\to\infty} r\tilde{u}'(r) > 0$ and this implies $\lim_{r\to\infty}\tilde{u}(r) = +\infty$.

If $\liminf_{r\to\infty}\beta(r)/r > 0$, there exist $c_1 > 0$ and $c_2 \in \mathbf{R}$ such that $w(x) \geqq c_1 u(x) + c_2$ a.e. Thus $\tilde{w}(r) \geqq c_1\tilde{u}(r) + c_2$, which contradicts $w\in L^1(\mathbf{R}^2)$. The case $D(\beta)$ bounded above is even simpler: $\tilde{u}\to\infty$ contradicts $u(x)\in D(\beta)$ a.e.

If $|\mathrm{grad}\, u| \in M^2(\mathbf{R}^2)$ we prove that $u\to -\infty$ in integral mean, a contradiction with $\tilde{u}(r)\to+\infty$: in fact since $u\in L^1_{\mathrm{loc}}(\mathbf{R}^2)$, $|\mathrm{grad}\, u|\in M^2(\mathbf{R}^2)$ and $ms\,[u>\lambda] < \infty$ for all $\lambda\in\mathbf{R}$ (for $u\to-\infty$ in measure) then

$$(2.2) \qquad \int (u-\lambda)^+ \leqq C\|\mathrm{grad}\, u\|_{M^2} ms[u>\lambda]$$

where C is independent of u and λ (see lemma 5.9, [2]).

Hence

$$\int_{B_R} u \leqq \int_{\mathbf{R}^2} (u-\lambda)^+ + \lambda\,|B_R| \leqq C\|\operatorname{grad} u\|_{M^2}\, ms[u>\lambda] + \lambda\,|B_R|.$$

Letting $\lambda \to -\infty$ we conclude that $|B_R|^{-1}\int_{B_R} u(x)dx \to -\infty$ as $R\to\infty$. #

Corollary 2.1. *If $f\in L^1(\mathbf{R}^2)$ and $\int f \leqq 0$, (E_f) cannot have a solution if* (i) *or* (ii) *of Proposition* 2.1 *holds.*

Proof. If u is a solution and $w=\Delta u + f \in \beta(u)$ a.e., since $w(x)>0$ a.e., $\int w > 0$. Therefore $\int \Delta u > 0$. #

Condition (i) is quoted in [2] to conclude the non-existence of solutions if $\int f < 0$. There the problem of finding solutions for f such that $\int f > 0$ is posed, thus motivating our work.

We comment now on the appearance of condition $|\operatorname{grad} u| \in M^2(\mathbf{R}^2)$ for solutions of (E). First, if $u\in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2)$ is a radial function and $\Delta u\in L^1(\mathbf{R}^2)$ then (2.1) implies that $ru'(r)$ is bounded (we write $u=u(r)$, with $r=|x|>0$) and thus $|u'(r)| = |\operatorname{grad} u| \in M^2(\mathbf{R}^2)$ (see lemma A.3, [2]). More generally if $u\in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2)$, $\Delta u\in L^1(\mathbf{R}^2)$ then $|\operatorname{grad} u|\in M^2(\mathbf{R}^2)$ is equivalent to $\operatorname{grad} u\to 0$ in a.m. (lemma A.11, [2]) and this is the case if, for instance, $u\in L^p(\mathbf{R}^N)$, $1\leqq p\leqq\infty$ (lemmas A.14, A.15, [2]).

In particular, $|\operatorname{grad} u|\in M^2(\mathbf{R}^2)$ for the solutions of [2]. We are going to obtain solutions for (P) as limits of those of [2] and the condition will hold in the limit.

The previous discussion leads us to pose our problem from $V = L^1_+(\mathbf{R}^2) = \{f\in L^1(\mathbf{R}^N) : \int f > 0\}$ into the class

$$\mathfrak{L} = \left\{ u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2) : |\operatorname{grad} u| \in M^2(\mathbf{R}^2),\ \Delta u\in L^1(\mathbf{R}^2),\ \int \Delta u = 0 \right\}.$$

Define $L^1(\mathbf{R}^2)_r$, the set of integrable radial functions on $\mathbf{R}^2$, as the completion of $C^1(\mathbf{R}^2)_r$ in L^1-norm. For $f\in L^1_+(\mathbf{R}^2)_r$ we shall consider radial solutions, i.e. $u\in\mathfrak{L}_r := L^1_{\mathrm{loc}}(\mathbf{R}^2)_r \cap \mathfrak{L}$. $u\in\mathfrak{L}_r$ if and only if it is representable as $u(r)\in C^1(]0,\infty[)$ with $(d/dr)(rdu/dr)\in L^1(0,\infty)$ and $\lim_{r\to\infty} ru'(r)=0$.

The next theorem is our main result stated in full detail:

Theorem 2.1. *The following properties are equivalent:*

(i) *There is an $a\in\mathbf{R}$ for which $(-\infty,a)\subset D(\beta)$ and*

$$\int_{-\infty}^{a} \beta(t)e^{-bt}dt < \infty \qquad \text{for every } b>0$$

(condition B_2*).*

(ii) *There exists some* $f \in L^1(\mathbf{R}^2)_r$ *for which* (P_f) *admits a radial solution.*
(iii) (P) *is well-posed from* $L^1_+(\mathbf{R}^2)$ *into* $\mathfrak{L}$.

PROOF. We shall proceed in this order: (i) $\Leftrightarrow$ (ii), (ii) $\Leftrightarrow$ (iii), as in [6].

(ii) $\Rightarrow$ (i): Let $u \in \mathfrak{L}_r$ be a solution of (P_f) and set $w = \Delta u + f$. As $ru'(r) \to 0$ when $r \to \infty$, for every $\varepsilon > 0$ there exist $C \in \mathbf{R}$ and $r_0 > 0$ such that if $r > r_0$

$$u(r) = u(r_0) + \int_{r_0}^{r} u'(r)dr \geqq u(r_0) - \varepsilon \lg(r/r_0) = C - \varepsilon \lg r.$$

Then

$$\infty > \int_{r_0}^{\infty} w(r)rdr \geqq \int_{r_0}^{\infty} \beta^0(C - \varepsilon \lg r)rdr = K \int_{-\infty}^{t_0} \beta(t)\exp(-2t/\varepsilon)dt$$

with $t = C - \varepsilon \lg r$, $t_0 = C - \varepsilon \lg r_0$, $K = (1/\varepsilon)\exp(2C/\varepsilon) > 0$. As ε is arbitrary > 0, condition (B_2) follows.

(i) $\Rightarrow$ (ii): Assume that a continuous differentiable function $g : (-\infty, 0) \to \mathbf{R}$ is given such that

$$g \geqq 1, \quad g \text{ is decreasing}, \quad \lim_{s \to -\infty} g(s) = \infty. \tag{2.3}$$

Define u by

$$\begin{cases} g(u)\dfrac{du}{dr} = -\dfrac{1}{r}, & \text{if } r > 1, \\ u(1) = 0. \end{cases} \tag{2.4}$$

If $g \in C^k$, $k \geqq 0$, (2.4) has a unique solution $u \in C^{k+1}$ defined in $(1, \infty)$ by $G(u) := \int_0^u g(s)ds = -\lg r$. u is decreasing, $ru'(r)$ is negative and increasing. Also $\lim_{r\to\infty} u(r) = -\infty$, for if on the contrary $u(r) \geqq C > -\infty$ for $r > 1$ we should have $g(u) \leqq g(C)$ and

$$u(1) - u(r) = \int_1^r -u'(s)ds \geqq \int_1^r \frac{ds}{sg(C)} = \frac{1}{g(C)} \lg r$$

so that $u(r) \to -\infty$. As we have assumed $\lim_{s\to-\infty} g(s) = \infty$ it follows from (2.4) that $\lim_{r\to\infty} ru'(r) = 0$. As ru' is increasing $\Delta u(x) = (1/r)(ru')' \geqq 0$ a.e. and we conclude that $\Delta u \in L^1(|x| > 1)$ (interpret $u = u(r)$ as a radial function in $\mathbf{R}^2$).

Set $w(r) = \beta^2(u(r))$ for $r \geqq 1$. We have for $\int_{|x|>1} w$:

$$\int_{|x|>1} w(x)dx = 2\pi\int_1^\infty w(r)rdr = 2\pi\int_1^\infty \beta^0(u(r))rdr = 2\pi\int_{-\infty}^0 \beta^0(u)g(u)r^2(u)du$$

$$= \{\text{since } r = \exp(-G(u))\} \qquad 2\pi\int_{-\infty}^0 \beta^0(u)g(u)\exp(-2G(u))du = (*).$$

We shall find a g satisfying (2.3) and such that $(*)$ is finite. Then continuing u smoothly and radially to all of $\mathbf{R}^2$, u will be a radial solution of (P_f) if we set $f := -\Delta u + w$.

To build such a g take a sequence $\{a_n\}$ with $a_n < a_{n-1} < 0$ and $\int_{-\infty}^{a_n} \beta(t)\exp(-4\sqrt{n}t)dt < n^{-2}$. Take now g satisfying (2.3) and such that $g(a_n) \leqq \sqrt{n}$. Then $-G(u) = \int_u^0 g(s)ds \leqq g(a_n)|u| \leqq \sqrt{n}|u|$ if $a_n < u < 0$ and

$$\int_{-\infty}^0 \beta^0(u)g(u)\exp(-2G(u))du$$

$$= \sum_{n=1}^\infty \int_{a_{n+1}}^{a_n} \beta^0(u)g(u)\exp(-2G(u))du + \int_{a_1}^0 \beta^0(u)g(u)\exp(-2G(u))du$$

$$\leqq \text{finite} + \sum_n \sqrt{n+1}\int_{a_{n+1}}^{a_n} \beta^0(u)\exp(2\sqrt{n}u)du$$

$$\leqq \text{finite} + \sum_n \sqrt{n+1}/n^2 = \text{finite}. \qquad \#$$

It remains to prove (ii) ⇔ (iii). We arrange the proof of this more difficult part in a series of lemmas, obtaining at the same time the main properties of the solutions.

LEMMA 2.1. *If we can solve* (P) *in* $\mathfrak{L}_r$ *for a radial* f *with* $\int f > 0$, *we can solve* (P) *in* $\mathfrak{L}_r$ *for a radial* f *with* $\int f$ *arbitrarily small* (*and positive*).

PROOF. Let $u \in \mathfrak{L}_r$ be a solution of (P_f), $f \in L^1_+(\mathbf{R}^2)_r$. Set $u_c = u - c$ for $c > 0$. Then $u_c \in \mathfrak{L}_r$, $\Delta u_c = \Delta u$ so that $\int \Delta u_c = 0$. Set $w_c = \beta^0(u_c)$: $\{w_c\}$ is a monotone sequence, $w_c \to 0$ a.e. as $c \to \infty$, hence $w_c \to 0$ in $L^1(\mathbf{R}^2)$. Put $f_c = -\Delta u_c + w_c$, $f_c \in L^1_+(\mathbf{R}^2)_r$. As $c \to \infty$, $\int f_c \downarrow 0$. #

Now we consider the process by which solutions are going to be obtained for general $f \in L^1_+(\mathbf{R}^2)$. The idea is to replace (P) by an approximate problem $(P)^\lambda$ by modifying β in such a manner that [2] applies:

For each $\lambda : 0 < \lambda < \sup \beta(\mathbf{R})$ take $r_\lambda \in D(\beta)$ such that $\lambda \in \beta(r_\lambda)$. Set $\beta^\lambda(s) = \beta(s + r_\lambda) - \lambda$. Then $0 \in \beta^\lambda(0) \cap \operatorname{Int} \beta^\lambda(\mathbf{R})$ so that [2], theorem 3.1 solves the problem

$$(P)^\lambda \qquad -\Delta v_\lambda + \beta^\lambda(v_\lambda) \ni f$$

in the class $\mathfrak{L}$. Set $u_\lambda = v_\lambda + r_\lambda$. We want to find a solution u for $(P_{\beta f})$ as the limit of u_λ as $\lambda \downarrow 0$. In that direction we have

LEMMA 2.2. *As* $\lambda \downarrow 0$ *the sequences* $\{u_\lambda\}$ *and* $\{W_\lambda\}$, *where* $W_\lambda = f + \lambda + \Delta u_\lambda \in \beta(u_\lambda)$, *are nonincreasing.*

PROOF. Take $0 < \lambda < \eta$. Then $-\infty < r_\lambda \leqq r_\eta$, $\Delta(u_\lambda - u_\eta) = W_\lambda - W_\eta + (\eta - \lambda)$ and $v \equiv v_\lambda - v_\eta = u_\lambda - u_\eta - (r_\lambda - r_\eta) \in \mathfrak{L}$. Lemma A.13 [2] implies that for any $p \in \mathfrak{P}$

$$\int \Delta v \cdot p(v) + \int |\operatorname{grad} v|^2 p'(v) \leqq 0, \tag{2.5}$$

$$\int [W_\lambda - W_\eta - (\lambda - \eta)] p(v_\lambda - v_\eta) + \int |\operatorname{grad}(v_\lambda - v_\eta)|^2 p'(v_\lambda - v_\eta) \leqq 0.$$

Now choose $\phi \in \mathfrak{P}$ with $\phi(s) = 0$ for $s \leqq 0$, $\phi'(s) > 0$ for $s > 0$ and put $p(s) = \phi(s + r_\lambda - r_\eta)$. Then

$$\int (W_\lambda - W_\eta + \eta - \lambda)\phi(u_\lambda - u_\eta) + \int |\operatorname{grad}(u_\lambda - u_\eta)|^2 \phi'(u_\lambda - u_\eta) \leqq 0.$$

From the nonnegativity of $(W_\lambda - W_\eta)\phi(u_\lambda - u_\eta)$ and $\eta > \lambda$ we conclude that $u_\lambda \leqq u_\eta$ a.e.

If β is continuous we conclude from $W_\lambda \in \beta(u_\lambda)$, $W_\eta \in \beta(u_\eta)$ that $W_\lambda \leqq W_\eta$ a.e. In general this argument only gives $W_\lambda \leqq W_\eta$ a.e. on $[u_\lambda < u_\eta]$. To get $W_\lambda \leqq W_\eta$ a.e. on $[u_\lambda = u_\eta]$ we apply Kato's inequality [9] to $u_\eta - u_\lambda$:

$$\Delta(u_\eta - u_\lambda)^+ \geqq \operatorname{sign}_0^+(u_\eta - u_\lambda) \cdot \Delta(u_\eta - u_\lambda).^\dagger$$

Since $(u_\eta - u_\lambda)^+ = (u_\eta - u_\lambda)$ we have a.e. on $[u_\eta = u_\lambda]$ that $\Delta(u_\eta - u_\lambda) \geqq 0$. #

LEMMA 2.3. *If* (P_f) *has a solution* $u \in \mathfrak{L}$ *and* $w = \Delta u + f$, *then* $u_\lambda \downarrow u$ *and* $W_\lambda \downarrow w$ *a.e. Thus u and w are unique. For* $j \in \mathfrak{J}_0$

$$\int j(w) \leqq \int j(f).$$

Also if $\hat{u}$, $\hat{w}$ *correspond to* $\hat{f}$

$$\int (w - \hat{w})^+ \leqq \int (f - \hat{f})^+$$

and

$$\|\operatorname{grad}(u - \hat{u})\|_{M^2} \leqq C \|f - \hat{f}\|_1, \qquad \textit{for a } C > 0.$$

† $\operatorname{sign}_0^+(s) = 0$ if $s \leqq 0$, $\operatorname{sign}_0^+(s) = 1$ if $s > 0$.

PROOF. Set $v = u - u_\lambda$. Formula (2.5) is valid for $p \in \mathfrak{P}_+$ since we have $v \in W^{1,1}_{loc}(\mathbf{R}^2)$, $|\operatorname{grad} v| \in M^2(\mathbf{R}^2)$, $\Delta v \in L^1(\mathbf{R}^2)$ and $ms[v > k] < \infty$ for $k > 0$ (but $ms[v < -k]$ is infinite). We conclude that $u_\lambda \geqq u$ a.e. and $W_\lambda \geqq w$ a.e. as before. In the limit we obtain $u_\infty = \lim u_\lambda \geqq u$ a.e. Since $\|\nabla u_\lambda\|_{M^2} \leqq C\|f\|_1$ we conclude that $\operatorname{grad} u_\lambda \to \operatorname{grad} u_\infty$ weakly in $[L^1_{loc}(\mathbf{R}^2)]^2$ and that $\|\operatorname{grad} u_\infty\|_{M^2} \leqq C\|f\|_1$ ($\|\cdot\|_{M^2}$ is l.s. continuous under weak L^1_{loc}-limits).

Set $w_\lambda = W_\lambda - \lambda \in \beta^\lambda(v_\lambda)$. From [2], p. 527, we conclude that $\{w_\lambda\}$ is relatively compact in $L^1_{loc}(\mathbf{R}^2)$ and $\|w_\lambda\|_1 \leqq \|f\|_1$. Therefore there exists $w_\infty = \lim_{\lambda \downarrow 0} w_\lambda$, limit in $L^1_{loc}(\mathbf{R}^2)$ and a.e., and $\|w_\infty\|_1 \leqq \|f\|_1$. Also $W_\lambda = w_\lambda + \lambda \to w_\infty$ in $L^1_{loc}(\mathbf{R}^2)$. We conclude that $W_\lambda \downarrow w_\infty \geqq w$. Passing to the limit in $(P)^\lambda$ we obtain $-\Delta u_\infty + w_\infty = f$. As $u_\lambda \to u_\infty$ a.e. and $W_\lambda \to w_\infty$ in $L^1_{loc}(\mathbf{R}^2)$ we have $w_\infty \in \beta(u_\infty)$.

We prove next that $\int \Delta u_\infty = 0$: Since $u \in \mathfrak{L}$, $\int \Delta u = 0$, $\lim_{r\to\infty} r u'(r) = 0$, i.e. for every $\varepsilon > 0$ there exists C, R such that $\tilde{u}(r) \geqq C - \varepsilon \lg r$ for $r > R$. As $u_\infty \geqq u$ we have $\lim_{r\to\infty} r\tilde{u}'_\infty(r) \geqq 0$. But Proposition 2.1 implies that $\lim r\tilde{u}'_\infty(r) \leqq 0$, so finally $\int \Delta u_\infty = 0$. Thus u_∞ is another solution of (P) and $u_\infty \geqq u$. Also we have $w_\infty \geqq w$; as $\int \Delta u = \int \Delta u_\infty = 0$, $\int f = \int w = \int w_\infty$, hence $w = w_\infty$ a.e. Finally if $\bar{v} = u_\infty - u$ then $v \in \mathfrak{L}$, $\bar{v} \geqq 0$ and $\Delta \bar{v} = 0$. It is easy to see (cf. [2], p. 533) that $\bar{v}$ must be constant, $u_\infty = u + c$. We have the following situation: $u \in W^{1,p}_{loc}(\mathbf{R}^2)$ for some $1 < p < 2$, $w(x) \in \beta(u(x)) \cap \beta(u(x) + c)$ a.e., $w \in L^1(\mathbf{R}^2)$. The proof of [2], lemma 3.5, applies to imply that either $w = 0$ (impossible) or $C = 0$. Thus $u_\infty = u$.

The final statements of the lemma are a consequence of the passage to the limit for they are valid for the solutions of [2], when $0 \in \beta(0) \cap \operatorname{Int} \beta(\mathbf{R})$. #

In particular, since (E) is invariant under rotations and solutions are unique, every possible solution of (P_f) for f radial must be radial. This implies the easy step (iii) ⇒ (ii) in the Theorem.

Next we use the approximations $(P)^\lambda$ to prove (ii) ⇒ (iii).

LEMMA 2.4. *If* (ii) *holds and* $f \in L^1_+(\mathbf{R}^2)$ *there exists a solution* u *of* (E) *with* $|\operatorname{grad} u| \in M^2(\mathbf{R}^2)$.

PROOF. Solve $(P_f)^\lambda$ as in Lemma 2.3 and try to pass to the limit as $\lambda \downarrow 0$. As $\{u_\lambda\}$ is nonincreasing we shall get a limit if we bound the sequence below. For that we use (ii): using Lemma 2.1 we take a radial $g \in L^1(\mathbf{R}^2)$ with $\int f > \int g > 0$ and such that (P_g) is solvable in $\mathfrak{L}_r$. We set

$$(2.6)\qquad \begin{cases} -\Delta u_\lambda + \beta(u_\lambda) - \lambda \ni f, & W_\lambda = f + \lambda + \Delta u_\lambda, \\ -\Delta \hat{u}_\lambda + \beta(\hat{u}_\lambda) - \lambda \ni g, & \hat{W}_\lambda = g + \lambda + \Delta \hat{u}_\lambda. \end{cases}$$

We use inequality (2.5) with $v = \hat{u}_\lambda - u_\lambda$ and $p \in \mathfrak{P}_+$, $p(s) = 1$ for $s > 0$. Then

$$
\begin{aligned}
0 &\leqq \int (\hat{W}_\lambda - W_\lambda) p(\hat{u}_\lambda - u_\lambda) + \int p'(\hat{u}_\lambda - u_\lambda)|\operatorname{grad}(\hat{u}_\lambda - u_\lambda)|^2 \\
&\leqq \int (g-f) p(\hat{u}_\lambda - u_\lambda).
\end{aligned} \tag{2.7}
$$

As $\hat{u}_\lambda$ converges a.e. to the solution $\hat{u}$ of (P_g), that is finite a.e., if $\lim_{\lambda\downarrow 0} u_\lambda < \hat{u}$ a.e. the Dominated Convergence Theorem implies that $\int (g-f) \geqq 0$; false. Thus we have $u = \lim_{\lambda\downarrow 0} u_\lambda \geqq \hat{u}$ on a set of positive measure. Since u_λ is nonincreasing as $\lambda \downarrow 0$ there is a k such that for every large $R > 0$, $ms\{[|u_\lambda| < k] \cap B_R(0)\}$ is greater than 0 uniformly in λ. Also $\{\operatorname{grad} u_\lambda\}$ is uniformly bounded in M^2-norm, so that lemma A.16 [2] implies that $\{u_\lambda\}$ is bounded in $L^{p^*}_{\mathrm{loc}}(\mathbf{R}^2)$ for $p^* = p/(2-p)$, $1 < p < 2$, and its limit $u \in L^{p^*}_{\mathrm{loc}}(\mathbf{R}^2)$.

Now repeat the argument of Lemma 2.3 to show that u is a solution. #

If we show that the solution of (E) found in Lemma 2.4 satisfies $\int \Delta u = 0$ the proof of (ii) $\Rightarrow$ (iii) will be complete. Recall that Proposition 2.1 implies that $\int \Delta u \leqq 0$.

LEMMA 2.5. *If* (ii) *holds then* (P_f) *is solvable in* $\mathfrak{L}_r$ *for every radial* $f \in L^1_+(\mathbf{R}^2)$.

PROOF. If $v \in \mathfrak{L}_r$ and $p \in \mathfrak{P}$ it is trivial to show that $rp'(v(r))v'(r)^2 \in L^1(0,\infty)$ and

$$
\int_r^\infty p(v)(rv')'dr + \int_r^\infty p'(v)v'(r)^2 r\,dr = -p(v)rv'(r) \tag{2.8}
$$

(compare with (2.5)). Apply (2.8) to $v = \hat{u}_\lambda - u_\lambda$ of (2.6) to get

$$
\begin{aligned}
\int_r^\infty (\hat{W}_\lambda - W_\lambda) p(\hat{u}_\lambda - u_\lambda) r\,dr + \int_r^\infty p'(\hat{u}_\lambda - u_\lambda)(\hat{u}'_\lambda - u'_\lambda) r\,dr + p(\hat{u}_\lambda - u_\lambda) r(\hat{u}'_\lambda - u'_\lambda) \\
\leqq \int_r^\infty |f-g|\, r\,dr.
\end{aligned}
$$

So

$$
r(\hat{u}'_\lambda - u'_\lambda) p(\hat{u}_\lambda - u_\lambda) \leqq \int_r^\infty |f-g|\, r\,dr.
$$

As $[r(u'_\lambda - u'_\lambda)]'$ converges in $L^1_{\mathrm{loc}}(0,\infty)$ ($\Delta(\hat{u}_\lambda - u_\lambda)$ converges in $L^1_{\mathrm{loc}}(\mathbf{R}^2)$) and $r(\hat{u}'_\lambda - u'_\lambda)$ is uniformly bounded in λ and r, we have $r(\hat{u}'_\lambda - u'_\lambda) \to r(\hat{u}' - u')$ a.e. and

$$
r(\hat{u}' - u') p(\hat{u} - u) \leqq \int_r^\infty |f-g|\, r\,dr \to 0 \qquad \text{as } r \to \infty. \tag{2.9}
$$

Assume now that $\int \Delta u = c_1 < 0$. Then $\lim_{r\to\infty} ru'(r) = c_1 < 0$ so that $u(r) \sim c_1 \lg r$ for all large r. Since $\int \Delta \hat{u} = 0$, $r\hat{u}'(r) \to 0$ as $r \to \infty$, so that $\lim_{r\to\infty} \hat{u}(r)/\lg r = 0$ and $\hat{u} - u > 0$ for all large r. Taking in (2.9) $p \in \mathfrak{P}_+$ with $p(s) = 1$ for $s \geqq 0$ and letting $r \to \infty$ we conclude that $\lim r(\hat{u}' - u') \leqq 0$, contradicting the assumption.

#

Another step is provided by

LEMMA 2.6. *If* (ii) *holds then* (P_f) *is solvable in* $\mathfrak{L}$ *for every* $f \in L^1_+(\mathbf{R}^2)$ *with compact support in* $\mathbf{R}^2$.

PROOF. Suppose that $\mathrm{supp}(f) \subset B_R(0)$. By Lemma 2.4 we obtain $\lim_{\lambda\downarrow 0} u_\lambda = u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2)$. In $[|x| > R]$ we have $\Delta u_\lambda \in \beta(u_\lambda) - \lambda$, $\Delta u \in \beta(u)$. The argument in [2], p. 542, shows that $u_\lambda \in L^\infty_{\mathrm{loc}}(|x| > R)$. Also, it can be shown that $v_\lambda = u_\lambda - r_\lambda$ tends to zero uniformly as $|x| \to \infty$ at any λ such that $\beta^{-1}(\lambda) = \{r_\lambda\}$ (see for instance [12], lemma 4). Thus for such λ and R_1 large, $W_\lambda \in \beta(u_\lambda)$ is bounded in $[|x| > R_1]$. As W_λ is nonincreasing in λ and $W_\lambda \geqq 0$, we have $w = \lim W_\lambda \in L^\infty[|x| > R_1]$ and as in [2] we conclude that $u \in C^1(|x| > R)$. Take $R_0 > R_1$ and C such that $u(x) > C$ for $|x| = R_0$.

We build now a radial comparison function $v \in \mathfrak{L}_r$, $v \leqq C$ on $[|x| = R_0]$ and such that $-\Delta v + \beta(v) \ni g$ for a $g \in L^1[|x| > R_1]$, $g \leqq 0$. In fact if $\bar{g}$ is an integrable radial function with support in $B_{R_1}(0)$ and $\bar{v}$ is the solution of (P_g), put $v = \bar{v} - k$, k a sufficiently large constant. Compare now u and v in $[|x| > R_0]$ to conclude that $u \geqq v$. As $\lim_{r\to\infty} rv'(r) = 0$ we have $\int \Delta u = \lim_{r\to\infty} r\bar{u}'(r) \geqq 0$. (To compare u and v begin by comparing u_λ and v_λ as in [2], then pass to the limit.) #

LEMMA 2.7. *If* $f_n \to f$ *in* $L^1_+(\mathbf{R}^2)$ *and* (P_{f_n}) *is solvable, then* (P_f) *is solvable.*

PROOF. Set

$$\begin{cases} -\Delta u^n_\lambda + \beta(u^n_\lambda) - \lambda \ni f_n, & w^n_\lambda = \Delta u^n_\lambda + f_n, \\ -\Delta u^n + \beta(u^n) \ni f_n, & w^n = \Delta u^n + f_n; \end{cases}$$

$$\begin{cases} -\Delta u_\lambda + \beta(u_\lambda) - \lambda \ni f, & w_\lambda = \Delta u_\lambda + f, \\ -\Delta u + \beta(u) \ni f, & w = \Delta u + f \quad \text{(as in Lemma 2.4)}. \end{cases}$$

We have $\int w^n_\lambda = \int f_n = \int w^n$. Also $\| w^n_\lambda - w_\lambda \|_1 \leqq \| f - f_n \|_1$. But $w^n_\lambda - w_\lambda \to w^n - w$ in $L^1_{\mathrm{loc}}(\mathbf{R}^2)$ and a.e. By Fatou we have $\| w^n - w \|_1 \leqq \| f - f_n \|_1$.

$$\left| \int w - f \right| \leqq \int |w - w^n| + \left| \int w^n - f_n \right| + \int |f_n - f| \leqq 2\| f_n - f \|_1 \to 0$$

as $n \to \infty$. Therefore $\int w = \int f$, $\int \Delta u = 0$. #

Thus the proof of (ii) $\Rightarrow$ (iii) and of the Theorem is complete. We record the main properties of the solutions of (P):

PROPOSITION 2.2. *The following holds for the solutions of* (P) *in the class* $\mathfrak{L}$:

(i) $G: f \mapsto u$ *is a single-valued order preserving map from* $L^1_+(\mathbf{R}^2)$ *into* $W^{1,p}_{loc}(\mathbf{R}^2)$, $1 \leqq p < 2$. *Every solution* $u \in \mathfrak{L}$ *converges to* $-\infty$ *at infinity in measure, in integral mean and in angular mean. G is bounded on any weakly relatively compact subset of* $L^1_a(\mathbf{R}^2) = \{f \in L^1(\mathbf{R}^2) : \int f \geqq a\}$, $a > 0$, *and hence it is weakly sequentially continuous.*

(ii) $T: f \rightarrow w$ *is an ordered contraction of* $L^1_+(\mathbf{R}^2)$ *invariant under rotations and translations.*

(iii) *If* $j \in \mathfrak{J}_0$ *and* $f \in L^1_+(\mathbf{R}^2)$, $w = Tf$:

$$\int j(w) \leqq \int j(f).$$

In particular $\|w\|_p \leqq \|f\|_p$, $1 \leqq p < \infty$, $\sup w \leqq \sup f$.[†]

(iv) *There is a constant* $C \leqq \|\operatorname{grad} E_2\|_{M^2}$ *such that*

$$\|\operatorname{grad}(u - \hat{u})\|_{M^2} \leqq C \|f - \hat{f}\|_1.$$

PROOF. Most of the properties are consequences of theorem 3.1, [2] after passing to the limit. The convergence of u is a consequence of Lemma 2.8 to follow. Let us prove the boundedness of G: Let F be a subset of $L^1_+(\mathbf{R}^2)$ such that, uniformly in $f \in F$, (i) $\int f \geqq a$, (ii) there exists a constant C_1 such that $\int |f| \leqq C_1$, (iii) for every $\varepsilon > 0$ there exists a compact $K \subset \mathbf{R}^2$ such that $\int_{\mathbf{R}^2 - K} |f| < \varepsilon$, (iv) for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\Omega \subset \mathbf{R}^2$ and $ms(\Omega) < \delta$ then $\int_\Omega |f| < \varepsilon$.

We repeat the proof of Lemma 2.6. Letting $\lambda \downarrow 0$ and $p(s) \rightarrow \operatorname{sign}_0(s)$[††] we obtain from (2.7) with $u = u_f$

$$\int_{[u_f > v]} (f - g) \geqq \int_{[u_f < v]} (f - g).$$

If $f \in F$ and $\int |g| < a/3$, we conclude that $\int_{[u_f \geqq u]} f > a/3$. This implies that $ms[u_f \geqq v] \leqq \delta$ for a certain $\delta > 0$. On the other hand $ms[u_f > \lambda] \leqq \beta^0(\lambda)^{-1} \int w_f \leqq \beta^0(\lambda)^{-1} \|f\|_1$. Thus on any sufficiently large ball B there exists $\lambda > 0$ such that $ms[|u_f| > \lambda] \equiv \sigma > 0$, σ independent of $f \in F$. Then lemma A.16 [2] implies that $\{u_f\}$ is bounded in $L^{p^*}_{loc}(\mathbf{R}^2)$, $p^* < \infty$. #

LEMMA 2.8. *Let* β *be a m.m. graph such that* $\beta(s) > 0$ *for* $s > a$, $a \in \mathbf{R}$. *Let*

[†] $\inf w = 0$.

[††] $\operatorname{sign}_0(s) = 1$ if $s > 0$, 0 if $s = 0$, -1 if $s < 0$.

$u \in \mathfrak{L}$ be a solution of $-\Delta u + \beta(u) \ni f$, $f \in L^1(\mathbf{R}^2)$. *Then* $\limsup u \leqq a$ *in measure, in integral mean and in angular mean, as* $|x| \to \infty$.

PROOF. If $w = \Delta u + f \in L^1(\mathbf{R}^2)$, $w \in \beta(u)$ a.e. it is immediate that $\limsup_{x\to\infty} u \leqq a$ (m), i.e. $ms[u > \lambda] < \infty$ if $\lambda > a$.

For the convergence in mean apply (2.2) for $\lambda > a$:

$$\int_{B_R} |u| \leqq \int_{B_R} (u-\lambda)^+ + \lambda\,|B_R| \leqq C\|\operatorname{grad} u\|_{M^2}\, ms[u > \lambda] + \lambda\,|B_R|.$$

We conclude that $\limsup_{R\to\infty} |B_R|^{-1} \int_{B_R} u(x)dx \leqq a$.

For the angular convergence, suppose that $\limsup_{r\to\infty} \tilde{u}(r) = b > a$. Take a sequence $\{r_n\}$ with $r_{n+1} \geqq 2r_n$ and $\tilde{u}(r_n) \geqq b - \varepsilon$ for an ε, $b - a > 2\varepsilon > 0$. As $r\tilde{u}'(r) \to 0$ take $n(\varepsilon)$ such that for $n \geqq n(\varepsilon)$ and $r \geqq r_n$, $|r\tilde{u}'(r)| \leqq \varepsilon^2$. Now for $r_n \leqq r \leqq r_n/\varepsilon$ we have for ε small enough

$$|\tilde{u}(r) - \tilde{u}(r_n)| \leqq \int_{r_n}^{r} |u'(r)|\,dr \leqq \varepsilon^2 \lg(r/r_n) \leqq \varepsilon^2 \lg|\varepsilon| < \varepsilon$$

thus $u(r) > b - 2\varepsilon$. But now

$$\int_{[r_n \leqq |x| \leqq r_n/\varepsilon]} u(x)dx \geqq (b - 2\varepsilon)\, ms[r_n \leqq |x| \leqq r_n/\varepsilon].$$

This contradicts $\limsup_{R\to\infty} |B_R|^{-1} \int_{B_R} u \leqq a$. #

3. $N = 1, 2$. A general discussion

$N = 2$. We discuss here the two-dimensional problem

$$(\mathrm{P}_{\beta f}) \qquad -\Delta u + \beta(u) \ni f, \qquad \int \Delta u = 0.$$

We seek solutions for $f \in L^1(\mathbf{R}^2)$ in $\mathfrak{L} = \{u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2) : |\operatorname{grad} u| \in M^2(\mathbf{R}^2)$, $\int \Delta u = 0\}$. Several cases occur:

Case 1. $0 \in \operatorname{Int} \beta(\mathbf{R})$

By translation we may consider that $0 \in \beta(0) \cap \operatorname{Int} \beta(\mathbf{R})$ as in [2]. Theorem 3.1, [2] says that (P) is well-posed in $\mathfrak{L}_1 = \{u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2), |\operatorname{grad} u| \in M^2(\mathbf{R}^2)\}$. $\int \Delta u = 0$ comes as a consequence of $w \in L^1(\mathbf{R}^2)$, $w \in \beta(u)$ a.e. Thus (P) *is well-posed in* $\mathfrak{L}$.

Case 2. $0 \notin \beta(\mathbf{R})$

Consider only the case $\beta(\mathbf{R}) \subset (0, \infty)$. This is our Section 2. (P) *is well-posed from* $L^1_+(\mathbf{R}^2)$ *into* $\mathfrak{L}$ *if and only if* β *satisfies condition* (B_2) *at* $-\infty$.

Case 3. $\beta \equiv 0$

As $T: f \to w$ is the zero map, (P) *degenerates* from our point of view. $-\Delta u = f$ has solution (determined up to a constant) in BMO($\mathbf{R}^2$), the space of functions of bounded mean oscillation. If $\int f = 0$ they belong to $\mathfrak{L}$ by virtue of

LEMMA 3.1. *Let* $N \geqq 2$, $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$, $\Delta u \in L^1(\mathbf{R}^N)$ *and* $\operatorname{grad} u(x) \to 0$ *in absolute integral mean as* $x \to \infty$. *Then* $u = E_N * (-\Delta u) + c$ *for a certain* $c \in \mathbf{R}$.

PROOF. If $N \geqq 3$ we have the convolution $L^1(\mathbf{R}^N) * M^{N/(N-2)}(\mathbf{R}^N) \to M^{N/(N-2)}(\mathbf{R}^N)$, see appendix [2]. For $N = 2$ we have $L^1(\mathbf{R}^N) * \mathrm{BMO}(\mathbf{R}^2) \to \mathrm{BMO}(\mathbf{R}^2)$. See Appendix at the end of this work.

Thus set $\bar{u} = (-\Delta u) * E_N$. Then $|\operatorname{grad} \bar{u}| \in M^{N/(N-1)}(\mathbf{R}^N)$. Put $v_i = (\partial/\partial x_i)(u - \bar{u})$ for $i = 1, \cdots, N$: $v_i \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ and $\lim_{x \to \infty} v_i = 0$ (a.i.m.). Lemma A.8, [2] implies that $v_i = 0$, hence $u = \bar{u} + c$. #

Thus (P) *is well-posed from* $\{f \in L^1(\mathbf{R}^2) : \int f = 0\}$ *into* $\mathfrak{L}$.

Case 4. β non-positive or non-negative and $0 \in \beta(0)$

Consider, for instance, $\beta(\mathbf{R}) \subset [0, \infty)$ and $\beta^{-1}(0) = (-\infty, b]$, $b \in \mathbf{R}$. Then

(i) $\int f > 0$. Theorem 2.1 and Proposition 2.2 apply with minor changes; now we get $\limsup_{r \to \infty} \bar{u}(r) \leqq b$. If $f \geqq 0$ a.e., $f \neq 0$ then $u \geqq b$ a.e. Solutions are unique.

(ii) $\int f < 0$. No solution exists (see Corollary 2.1).

(iii) If $\int f = 0$, necessarily $w = \Delta u + f = 0$ a.e. for any solution ($\int w = \int f = 0$ and $w \geqq 0$). Thus we are reduced to solve $-\Delta u = f$ with $u \in \mathfrak{L}$ (case 3) and also u bounded above. This last condition is fulfilled if $f \in L^p(\mathbf{R}^2)$, $1 < p \leqq \infty$, f has compact support (and $\int f = 0$): then $f * E_2$ is a continuous solution of (P) that converges to zero at ∞ (uniformly, $u(x) = O(|x|^{-1})$). On the other hand take $f \in L^1(\mathbf{R}^2)_r$, $\int f = 0$ and $f(x) = (r \lg(1/r))^{-2}$ for all $r = |x|$ small. Then u is equivalent near 0 to $\lg(-\lg r)$, so that $u(0) = +\infty$.

(P) *is well-posed from a* V, $L^1_+(\mathbf{R}^2) \subset V \subset \{f \in L^1(\mathbf{R}^2); \int f \geqq 0\}$ *into* $\mathfrak{L}$. (For the well-posedness use the Lemma 2.3.)

$N = 1$. Here

$$(\mathrm{P}_{\beta,f}) \qquad -u'' + \beta(u) \ni f, \qquad u'(\pm\infty) = 0.$$

We take $\mathfrak{L} = \{u \in C^1(\mathbf{R}) : u'(\pm\infty) = 0\}$. The same cases as in $N = 2$ apply and similar phenomena occur. Case 1 is dealt with in [2]: (P) *is well-posed in* $\mathfrak{L}$. Case 2 for positive β is Crandall and Evans's [6]: (P) *is well-posed from* $L^1_+(\mathbf{R})$ *into* $\mathfrak{L}$ *if and only if* β *is integrable at* $-\infty$.

Case 3, $\beta \equiv 0$ has the explicit solution $u = c - x\int_{-\infty}^{x} f(s)ds + \int_0^x f(s)ds + \int_0^x f(s)sds$; $u \in \mathfrak{L}$ if and only if $\int f = 0$ and case 4 goes as in $N = 2$: thus if $\beta \geqq 0$ and $\int f = 0$ we are reduced to solve $-u'' = f$ with $u \in \mathfrak{L}$ bounded above. As $u \in C^1(\mathbf{R})$ non-boundedness can occur only at ∞. For instance, if f has compact support, u is constant outside $\operatorname{supp}(f)$ and (P_f) is solvable. On the contrary, if $f \in L^1(\mathbf{R})$, $\int f = 0$ and $f(x) = x^{-2}$ for large x, $u(x)$ is equivalent to $\lg x$ as $|x| \to \infty$ and (P_f) is not solvable.

4. $N \geqq 3$

The situation is quite different in dimensions $N \geqq 3$. Thus it is shown in [1] that we can impose the condition $\int \Delta u = 0$ on the solutions of [2] only if β satisfies

$$\lim_{r \to 0} \int_0^r \beta(r) |r|^{-2(N-1)/(N-2)} dr = \infty.$$

On the other hand, [2] shows solutions in $M^{N/(N-2)}(\mathbf{R}^N)$, hence they converge to 0 (a.i.m.). In fact lemma A.5 [2] says that for a $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $N \geqq 3$ with $\Delta u \in L^1(\mathbf{R}^N)$, $\lim_{x \to \infty} u = 0$ (a.i.m.) if and only if $u = E_N * (-\Delta u)$ and hence $u \in M^{N/(N-2)}(\mathbf{R}^N)$ and $|\operatorname{grad} u| \in M^{N/(N-1)}(\mathbf{R}^N)$.

Set $\mathfrak{L} = \{u \in L^1_{\mathrm{loc}}(\mathbf{R}^N) : |\operatorname{grad} u| \in M^{N/(N-1)}(\mathbf{R}^N)\}$ and $\mathfrak{L}_c = \{u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N): u - c \in M^{N/(N-2)}(\mathbf{R}^N)\}$, $\mathfrak{L}_c \subset \mathfrak{L}$. Bénilan and Brézis [1] have studied (P) = (E) in case $\beta(\mathbf{R}) \ni 0$. If $\beta^{-1}(0) = \{a\}$, (P) *is well-posed in* $\mathfrak{L}$ *and* $\mathfrak{L}_a$. But if $\beta^{-1}(0) = [a, b]$, $-\infty \leqq a \lneqq b \leqq \infty$, (P) *is well-posed in* $\mathfrak{L}_c$ *for every* $c \in \beta^{-1}(0)$; (P) *is well-posed in* $\mathfrak{L}$ *only if* $\beta \equiv 0$. Every solution in $\mathfrak{L}$ belongs to an $\mathfrak{L}_c$.

If $0 \notin \beta(\mathbf{R})$ we show a *negative* result: A reasonable well-posed problem is not possible as a consequence of the following

PROPOSITION 4.1. *Let* $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$ *with* $\operatorname{grad} u \to 0$ (a.i.m.) [*in particular if* $|\operatorname{grad} u| \in M^{N/(N-1)}(\mathbf{R}^N)$] *and* $\Delta u \in L^1(\mathbf{R}^N)$ *and let* β *be a m.m.g. such that* $\beta(\mathbf{R}) \subset (0, \infty)$. *Then* $\beta^0(u)$ *is not integrable on* $\mathbf{R}^N$.

PROOF. Under the conditions of the proposition, Lemma 3.1 implies that $u = E_N * (-\Delta u) + c$ for a certain $c \in \mathbf{R}$. Thus $\lim_{x \to \infty} u = c$ in measure. But $\beta^0(u) \in L^1(\mathbf{R}^N)$ and $\beta(\mathbf{R}) \subset (0, \infty)$ imply that $u \to -\infty$ in measure, a contradiction. #

Proposition 4.1 excludes, for instance, the possibility of any *radial* solution, for if $u = u(r)$ is a radial solution, $u \in C^1(0, \infty)$ and $u'(r)r^{N-1}$ is bounded. But this implies $|\operatorname{grad} u| = |u'(r)| \in M^{N/(N-1)}(\mathbf{R}^N)$ (see lemma A.3 [2]).

Appendix

We collect here a series of results, some of them well-known, related to the solution of $-\Delta u = f \in L^1(\mathbf{R}^2)$. We thank R. Moriyon for valuable advice.

LEMMA A.1. *The convolution $(f, g) \mapsto f * g$ is a continuous bilinear mapping $H^1(\mathbf{R}^N) \times L^1(\mathbf{R}^N) \to H^1(\mathbf{R}^N)$, where $N \geqq 1$ and H^1 denotes the Hardy space.*

PROOF. Let R_j, $j = 1, \cdots, N$, be the Riesz transforms, $R_j : L^1 \to \mathcal{S}'$ (the space of tempered distributions) defined by

$$R_j f(\xi) = \frac{\xi_j}{|\xi|} \hat{f}(\xi),$$

where $\wedge$ denotes Fourier transform. Since $H^1(\mathbf{R}^N) = \{f \in L^1(\mathbf{R}^N): R_j(f) \in L^1(\mathbf{R}^N),\ j = 1, \cdots, N\}$ is normed by $\|f\|_{H^1} = \|f\|_{L^1} + \Sigma_{j=1}^N \|R_j f\|_{L^1}$ and $R_j(f * g) = R_j f * g$ in $\mathcal{S}'$ (check the Fourier transforms), we deduce for $f \in H^1(\mathbf{R}^N)$, $g \in L^1(\mathbf{R}^N)$ that

$$\|f * g\|_{H^1} \leqq \|f\|_{H^1} \cdot \|g\|_{L^1}. \qquad \#$$

We recall that the members of the space $\mathrm{BMO}(\mathbf{R}^N)$ of "functions of bounded mean oscillation" are classes of functions up to an arbitrary additive constant. Denote by $[f]$ the class of f. We have

LEMMA A.2. *There is a continuous bilinear mapping* $\mathrm{BMO} \times L^1 \xrightarrow{\tilde{*}} \mathrm{BMO}$. *For $[f] \in \mathrm{BMO}(\mathbf{R}^N)$, $g \in L^1_0(\mathbf{R}^N)$ (integrable with compact support), $\tilde{*}$ is ordinary convolution (except for a constant). If $g_n \in L^1_0$ and $g_n \to g$ in $L^1(\mathbf{R}^N)$*

$$f \tilde{*} g(x) = \lim_{n \to \infty} \int_{\mathbf{R}^N} f(t) g_n(x - t) dt \qquad \text{(limit in BMO)}.$$

PROOF. Since $\mathrm{BMO}(\mathbf{R}^N)$ is the dual of $H^1(\mathbf{R}^N)$, we define $*$ by adjointness: for $[f] \in \mathrm{BMO}(\mathbf{R}^N)$, $g \in L^1(\mathbf{R}^N)$, $h \in H^1(\mathbf{R}^N)$ the following formula must hold:

$$\langle [f \tilde{*} g], h \rangle_{\mathrm{BMO} \times H^1} = \langle [f], g \tilde{*} h \rangle_{\mathrm{BMO} \times H^1}$$

where $\tilde{g}(s) = g(-s)$. So we have immediately

$$\|f \tilde{*} g\|_{\mathrm{BMO}} \leqq \|f\|_{\mathrm{BMO}} \cdot \|g\|_1.$$

If $g \in L^1_0(\mathbf{R}^N)$, the integral $f * g(x) = \int f(t) g(x - t) dt$ converges a.e. and belongs to $L^1_{\mathrm{loc}}(\mathbf{R}^N)$. By checking on the atoms of $H^1(\mathbf{R}^N)$ (compactly supported and bounded generators of H^1, see [11]) we conclude that $[f * g] = [f \tilde{*} g]$. For the final statement we use the continuity of $\tilde{*}$. #

LEMMA A.3. *If* $[f] \in \mathrm{BMO}(\mathbf{R}^N)$, $\nabla f \in M^p(\mathbf{R}^N)$ *and* $g \in L^1(\mathbf{R}^N)$. *Then* $\nabla(g \,\tilde{*}\, f) = g * \nabla f \in M^p(\mathbf{R}^N)$.

PROOF. Take first $g \in L_0^1(\mathbf{R}^N)$; the verification is immediate. For general $g \in L^1(\mathbf{R}^N)$, if $g_n \to g$ in L^1, then $f \,\tilde{*}\, g_n \to f \,\tilde{*}\, g$ in BMO and $\nabla f * g_n \to \nabla f * g$ in $L^1_{\mathrm{loc}}(\mathbf{R}^N)$ (for the convolution $M^p \times L^1 \to M^p$, see [2]). Thus we obtain the following

PROPOSITION A.1. *For every* $g \in L^1(\mathbf{R}^2)$ *there is a unique solution of* $-\Delta u = g$ *in* $\mathfrak{D}'(\mathbf{R}^2)$ *in the class* $\{u \in L^1_{\mathrm{loc}}(\mathbf{R}^N) : |\operatorname{grad} u| \in M^2(\mathbf{R}^2)\}$ *except for an arbitrary additive constant. It is given by*

$$u = \frac{1}{2\pi} \lg \frac{1}{|x|} \,\tilde{*}\, g \in \mathrm{BMO}(\mathbf{R}^2)$$

and then

$$\operatorname{grad} u = -\frac{1}{2\pi |x|} * g \in M^2(\mathbf{R}^2).$$

PROOF. Existence follows from the lemmas. Uniqueness was proved in Lemma 3.1. #

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